

5 Linear transformation

transformation

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function

||

mapping

$$f(x) = x^2 \xrightarrow{P_{2nd}} \frac{df}{dx} = 2x \xrightarrow{P_{1st}}$$

$$\frac{d}{dx} f \Rightarrow \text{1st degree}$$

↓
2nd degree

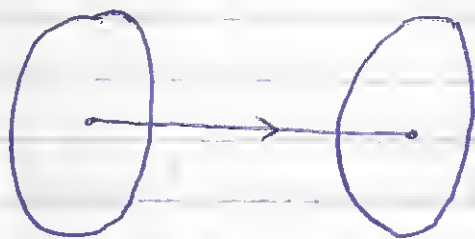
$$f(x) = x^2 \xrightarrow{P_2} P_3$$

$$\int x^2 dx = \frac{x^3}{3} + C$$

$$\int f(x) \Rightarrow g(x)$$

↓ ↓
2nd degree 3rd degree

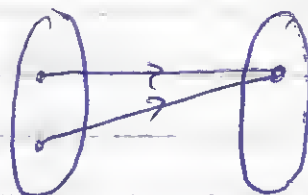
Function



Set(I)

Set(II)

or



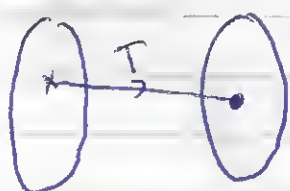
Domain

Codomain

Domain

Codomain/Range

Transformation

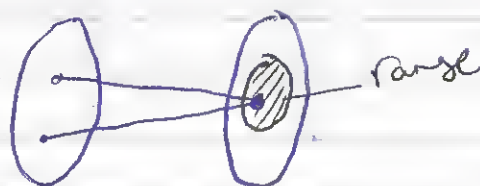


Domain
V

Codomain
W

or

$$T: V \rightarrow W$$



Domain

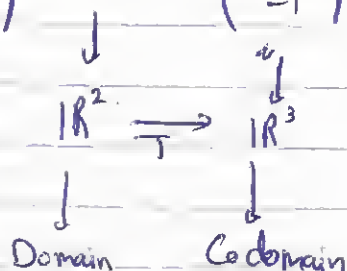
Codomain

range

Range is a subset of the codomain that contains the transformation of all points in the domain.

$$\text{Ex. } A = \begin{pmatrix} 1 & 0 \\ 2 & -1 \\ 3 & 4 \end{pmatrix}, V = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$AV = \begin{pmatrix} 1 & 0 \\ 2 & -1 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} = w$$



$$T(V) = w$$

$$\text{Such that } w = AV$$

Linear transformation: Let V and W to be vector spaces then the transformation $T: V \rightarrow W$ is called a linear transformation of V into W if the following two properties are true:

- 1- $T(\underline{u} + \underline{v}) = T(\underline{u}) + T(\underline{v})$
- 2- $T(c\underline{v}) = cT(\underline{v})$, where $c = \text{scalar}$

$$\text{Ex. } f(x) = \sin x$$

$$\sin(x_1 + x_2) \neq \sin x_1 + \sin x_2$$

$\therefore \sin x$ is not a linear transformation

$$\text{Ex. } f(x) = x^2$$

$$f(x_1 + x_2) = (x_1 + x_2)^2 \neq x_1^2 + x_2^2$$

$\therefore x^2$ is not a linear transformation

$$\text{Ex. } f(x) = x + 1$$

$$f(x_1) = x_1 + 1 \quad f(x_2) = x_2 + 1 \quad f(x_1 + x_2) = x_1 + x_2 + 1$$

$$f(x_1 + x_2) \neq f(x_1) + f(x_2)$$

$\therefore x + 1$ is not a linear transformation

Ex. Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, such that $T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2)$

find: ① The image of $v = (-1, 2)$

② The preimage of $w = (-1, 11)$

① $T(v) = T(-1, 2) = (-3, 3)$

② $v_1 - v_2 = -1$

$v_1 + 2v_2 = 11$

$-3v_2 = -12$

$v_2 = 4$

$v_1 = 3$

preimage $(3, 4)$

$T(3, 4) = (-1, 11)$

properties of linear transformation

1. $T(0) = 0$

Ex. let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$

such that $T(x_1, x_2) = (x_1 + 1, x_2)$

$T(0, 0) \neq (1, 0)$

\therefore Not a linear transformation

Matrix transformation

Def. Let A be an $m \times n$ matrix, then the transformation T is defined by $T(v) = Av$ is a linear transformation from \mathbb{R}^n to \mathbb{R}^m

Ex $A = \begin{pmatrix} 2 & -3 \\ -5 & 0 \\ 0 & -2 \end{pmatrix} \quad T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$

Ex $B = \begin{pmatrix} 1 & 0 & -1 & 2 \\ 3 & 1 & 0 & 0 \end{pmatrix} \quad T: \mathbb{R}^4 \rightarrow \mathbb{R}^2$

* Every $m \times n$ matrix represents a linear transformation from \mathbb{R}^n to \mathbb{R}^m and vice versa

ie. every linear transformation from \mathbb{R}^n to \mathbb{R}^m can be represented by an $m \times n$ matrix

The matrix $A = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$ represents a ~~trans~~

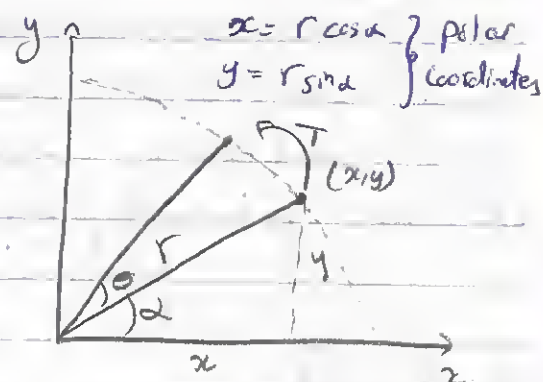
Transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ which has the property that it maps every vector in \mathbb{R}^2 counter-clockwise about the origin through an angle θ .

$$\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} r \cos \alpha \\ r \sin \alpha \end{pmatrix}$$

$$= \begin{pmatrix} r \cos \theta \cos \alpha - r \sin \theta \sin \alpha \\ r \sin \theta \cos \alpha + r \cos \theta \sin \alpha \end{pmatrix}$$

$$= \begin{pmatrix} r \cos(\theta + \alpha) \\ r \sin(\theta + \alpha) \end{pmatrix}$$

$A =$ rotation matrix

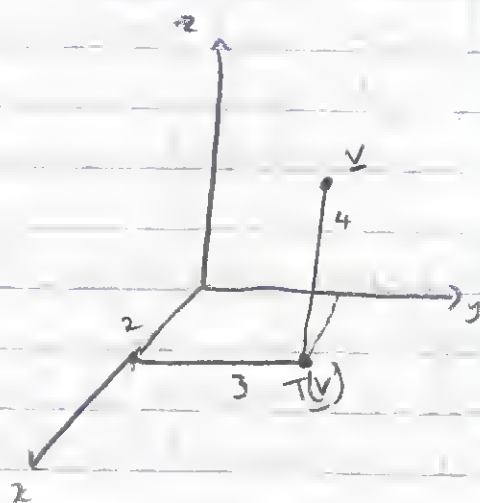


The matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ represents a transformation

$T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ which is called a projection in \mathbb{R}^3

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$$

A is a projection matrix



Ex. Let $A = \begin{pmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{pmatrix}$, $\underline{u} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, $\underline{b} = \begin{pmatrix} 3 \\ 2 \\ -5 \end{pmatrix}$, $\underline{c} = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}$

Which defines a transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ such that $T(\underline{x}) = A\underline{x}$.
Find: ① $T(\underline{u})$

① \underline{x} in \mathbb{R}^2 whose image is \underline{b}

② Is there more than one \underline{x} whose image is \underline{b} ?

③ Determine if \underline{c} is in the range of T .

① $T(\underline{u}) = \begin{pmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ -9 \end{pmatrix}$

② $A\underline{x} = \underline{b}$

$(A:\underline{b}) = \left(\begin{array}{cc|c} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & -5 \end{array} \right) \xrightarrow{\text{Echelon}} \left(\begin{array}{cc|c} 1 & 0 & 1.5 \\ 0 & 1 & -0.5 \\ 0 & 0 & 0 \end{array} \right) \quad \begin{matrix} x_1 = 1.5 \\ x_2 = -0.5 \end{matrix}$
 $\underline{x} = \begin{pmatrix} 1.5 \\ -0.5 \end{pmatrix}$

③ No, because there are no free variables \Rightarrow there is only 1 solution

④ $A\underline{x} = \underline{c}$

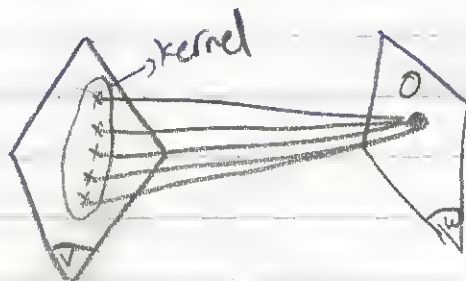
$\left(\begin{array}{cc|c} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & 5 \end{array} \right) \xrightarrow{\text{Echelon}} \left(\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -35 \end{array} \right)$

not consistent $\Rightarrow \underline{c}$ is not in the range of T

The kernel and range of linear transformation

Def of kernel

Let $T: V \rightarrow W$ be a linear transformation. Then the set of all vectors \underline{u} in V that satisfy $T(\underline{u}) = \underline{0}$ is called the kernel of T and is denoted by $\ker(T)$



Null space of a matrix

Null space of an $m \times n$ matrix A , written $\text{Nul}(A)$, is the set of all solutions of the homogeneous equation $A\underline{x} = \underline{0}$

$$\text{Nul}(A) = \{ \underline{x} : A\underline{x} = \underline{0} \}$$

Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a transformation given by $T(\underline{x}) = A\underline{x}$. Then the kernel of T is equal to the solution $A\underline{x} = \underline{0}$

Find the spanning set of the nullspace of the matrix

$$A = \begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix}$$

$$T: \mathbb{R}^5 \rightarrow \mathbb{R}^3$$

$$\left[\begin{array}{ccccc|c} -3 & 6 & -1 & 1 & -7 & 0 \\ 1 & -2 & 2 & 3 & -1 & 0 \\ 2 & -4 & 5 & 8 & -4 & 0 \end{array} \right] \xrightarrow{\text{echelon form}} \left[\begin{array}{ccccc|c} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

$$x_2 = t, \quad x_4 = s, \quad x_5 = r$$

$$x_1 = 2x_2 \quad -x_4 + 3x_5 = 0$$

$$x_3 + 2x_4 - 2x_5 = 0$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 2t + s - 3r \\ t \\ -2s + 2r \\ s \\ r \end{pmatrix} = t \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix} + r \begin{pmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{pmatrix}$$

$$\text{Spanning set } S = \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right\}$$